10/24 Recitation Solutions

1. Open the **Julia10Determinants** notebook. Complete"Worksheet Problem 1" in the notebook to visualize the 2×2 determinant as area.

   a) The area of the parallelogram is $|\det A| = |(1)(-4) - 2(2)| = 8$.

   b) Let's take the orthogonal matrix $Q = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$; we see that Q rotates the unit square. Since the image of the unit square is still a square of side length 1, the area has not changed; hence $\det Q = \pm 1$.

2. In class, we proved that the determinant of a diagonal matrix D is the product of the diagonal entries of D. Explain why this makes sense geometrically in the 3×3 case (by viewing the determinant as volume).

   Let $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T(\mathbf{v}) = D\mathbf{v}$. Let $C = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \,\middle|\, 0 \le x, y, z \le 1 \right\}$ be the unit cube in $\mathbb{R}^3$. Then the volume of $T(C)$ is $|\det D|$. Now $T(C)$ is the rectangular solid with sides given by the vectors

   $$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} d_1 \\ 0 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ d_2 \\ 0 \end{bmatrix}, \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ d_3 \end{bmatrix}.$$

   Hence its sides have length $|d_1|, |d_2|$, and $|d_3|$ and so it has volume $|d_1 d_2 d_3|$. It follows that $|\det D| = |d_1 d_2 d_3|$.

3. Show that the 2×2 determinant $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ satisfies the first two axioms of determinants discussed in class:

   - $\det I = 1$
   - row swaps negate the determinant

   Axiom 1: $\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1(1) - 0(0) = 1$.

   Axiom 2: $\det \begin{bmatrix} c & d \\ a & b \end{bmatrix} = bc - ad = -(ad - bc) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

4. Find the determinant of $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -3 & -1 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ by reducing to upper triangular form.

   Warning: we cannot simply use Julia to reduce this to Echelon form, since row swaps and scaling change the determinant. We must keep track of the row operations so that we can keep track of the determinant.

$$\begin{aligned}
\det \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -3 & -1 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 1 & 1 & 1 \end{bmatrix} &= \det \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & -1 & -5 & -6 \\ 0 & 0 & 0 & 4 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{(Replace R2 by R2-2R1)} \\
&= \det \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & -1 & -5 & -6 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & -4 & -5 \end{bmatrix} \quad \text{(Replace R4 by R4+R2)} \\
&= -\det \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & -1 & -5 & -6 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad \text{(Swap R3 and R4)} \\
&= -(1)(-1)(-4)(4) = -16
\end{aligned}$$

5. Open the **Julia10Determinants** notebook.
   (a) Complete "Worksheet Problem 2" in Julia to come up with formulas for the determinant of an $n \times n$ matrix, where $n = 2, 3, 4, 5, 6$. How long does each determinant calculation take?
   (b) Take a look at the 4×4 determinant formula you printed in Julia. Appreciate how tedious it would be to redo problem 3 using this formula. Then really appreciate how systematic it was to use row reduction instead.
   (c) For any n, you can write down a formula for the determinant of an $n \times n$ matrix (as you did for $n = 3, 4, 5, 6$). However, as you probably surmised, these formulas grow exponentially in n. We will see how to work with such formulas in lecture. While they are important for theoretical applications, they are not useful in doing concrete calculations. It is much more efficient to reduce to upper triangular form as in problem 3 because this process does not grow exponentially in n (and this is how computer programs compute determinants).